21-cm theory

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Abstract

It's not really useful to go through all the math to understand the theory behind the CD/EoR. However, it's nice to at least look at the steps once, in my opinion. Little did I know that the steps don't seem to be available anywhere on the internet. Why is there not a single textbook doing this? Unbelievable. In this document, I will tackle the painful but rewarding task of going through *all* the steps and providing Python code when needed in order to derive the final expression for the 21-cm brightness temperature as seen in most 21-cm theory papers.

1 Introduction

A little bit of motivation: we believe the 21-cm signal is a powerful probe of astrophysics and cosmology during the cosmic dawn (CD) of the first stars and galaxies and the subsequent epoch of reionisation (EoR). To find out whether it is actually detectable with current instruments, we need to calculate the strength or intensity of the 21-cm signal. That is the purpose of the calculations shown here. While the boxed equations are the ones seen in most 21-cm papers, it is the steps in between that are the focus here.

1.1 Setup - Radiative transfer

In order to describe the brightness or intensity of the 21-cm signal relative to the CMB i.e. $T_b - T_\gamma$, where $T_\gamma(z) = 2.73(1+z)$ K is the CMB temperature with $T_\gamma(0) = 2.73$ K, we begin with the basic radiative transfer equation describing photons emitted with specific intensity I_ν i.e. per unit frequency ν, passing through a gas cloud (i.e. the IGM) in the absence of scattering along path s (as shown in Figure 1):

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu, \tag{1}$$

where the first term describes absorption by the intervening gas while the second term describes emission. We need to describe how CMB photons interact with neutral hydrogen atoms to write out

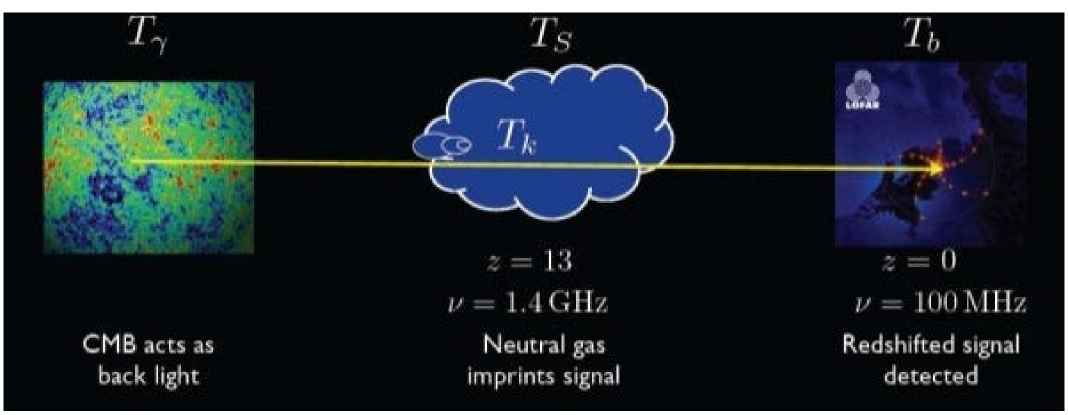

Figure 1: The CMB is essentially a backlight shining through the IGM composed primarily of neutral hydrogen before the EoR. The CMB photons interact with neutral hydrogen causing them to emit or absorb 21-cm photons.

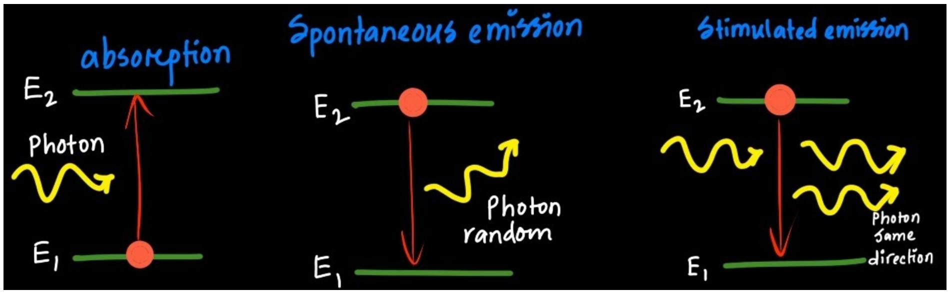

Figure 2: We consider three methods of interaction between CMB photons and neutral hydrogen atoms. The first one is absorption: the CMB photon hits a neutral hydrogen atom on the ground state (i.e. proton and electron spins are misaligned, E0) causing a spin-flip of the electron. As a result, the neutral hydrogen atom absorbs the CMB photon in favour of the excited state with aligned spins (E1). The second one is spontaneous emission: a neutral hydrogen in the excited state can spontaneously transition from E1 to E0 thus emitting a 21-cm photon. This process is very slow since the excited state has a very long lifetime ($\sim$ 11 million years). The third one is stimulated emission: a CMB photon (or from spontaneous emission) hits an excited hydrogen atom in E1, causing it to transition to the ground state and emit a 21-cm photon (with the original CMB photon unaffected).

the absorption coefficient α_ν and the emission coefficient j_ν. Note that the CMB photons must have a wavelength very near the 21-cm wavelength[1], otherwise they do not interact with the neutral hydrogen. This means that only a very small fraction of CMB photons can actually contribute to the 21-cm signal. In Figure 2, we describe the three interaction mechanisms we will consider: (i) absorption: the neutral hydrogen atom absorbs a 21-cm photon from the CMB promoting it from the ground state E0 to the excited state E1; (ii) spontaneous emission: while an excited hydrogen atom has a very long mean lifetime ($\sim$ 11 million years), it will eventually spontaneously emit a 21-cm photon and return to the ground state; (iii) stimulated emission: a 21-cm photon (from the CMB or from spontaneous emission) interacts with an excited hydrogen atom causing it to return to the ground state thus releasing an additional 21-cm photon. We can write the absorption coefficient as:

$$\alpha_\nu = n_1 B_{12} h\nu_{21}\phi(\nu), \tag{2}$$

where... Taking interaction mechanisms (ii) and (iii) into account, the emissivity j_ν can be written as:

$$j_\nu = n_2 h\nu_{21}\phi(\nu)A_{21} + n_2 h\nu_{21}\phi(\nu)B_{21}I_\nu, \tag{3}$$

where... $A_{21} = 2.85 \times 10^{-15}$ s $\sim 1/11$ million years is the rate of spontaneous decay of the upper level of the hyperfine transition.

Putting them back into Equation 1:

$$\begin{aligned}
\frac{dI_\nu}{ds} &= -\alpha_\nu I_\nu + j_\nu && (4)\\
&= -n_1 B_{12} h\nu_{21}\phi(\nu)I_\nu + n_2 h\nu_{21}\phi(\nu)A_{21} + n_2 h\nu_{21}\phi(\nu)B_{21}I_\nu && (5)\\
&= -\left(n_1 B_{12} - n_2 B_{21}\right) h\nu_{21}\phi(\nu)I_\nu + n_2 h\nu_{21}\phi(\nu)A_{21} && (6)\\
&\equiv -\alpha_\nu^{\text{eff}} I_\nu + j_\nu^{\text{eff}}, && (7)
\end{aligned}$$

where in the last line we define the effective absorption coefficient α_ν^{eff} that includes stimulated emission (think of it as "negative absorption"). Assuming the IGM gas is in equilibrium, we set $\frac{dI_\nu}{ds} = 0$ and simplify:

$$\begin{aligned}
0 &= n_2 h\nu_{21}\phi(\nu)A_{21} - \left(n_1 B_{12} - n_2 B_{21}\right) h\nu_{21}\phi(\nu)I_\nu && (8)\\
\left(n_1 B_{12} - n_2 B_{21}\right) I_\nu &= n_2 A_{21} && (9)
\end{aligned}$$

[1] Line width is not delta function, but very narrow...

We take $I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}(\exp(\frac{h\nu}{k_B T}) - 1)^{-1}$ as a blackbody following Planck's law (no Rayleigh-Jeans approximation for low frequencies yet!!). [2] Assuming matter and radiation in thermal equilibrium and therefore that $n_i \propto g_i \exp(-\frac{h\nu}{k_B T})$ is Boltzmann-distributed, we can derive the Einstein relations by plugging the Boltzmann distribution for n_1 and n_2 into Equation 8:

$$\boxed{A_{21} = \frac{2hv_{21}^3}{c^2} B_{21} \text{ and } B_{12} = \frac{g_2}{g_1} B_{21}} \tag{10}$$

The Einstein coefficients are fixed probabilities per time associated with each atom, and do not depend on the state of the gas of which the atoms are a part. Therefore, any relationship that we can derive between the coefficients at, say, thermodynamic equilibrium will be valid universally. We can therefore also assume the Einstein relations and plug them back into Equation 8:

$$n_2 A_{21} = (n_1 B_{12} - n_2 B_{21})\, I_\nu \tag{11}$$

$$n_2 \frac{2hv_{21}^3}{c^2} B_{21} = \left(n_1 \frac{g_2}{g_1} B_{21} - n_2 B_{21}\right) \frac{2h\nu_{21}^3}{c^2 \left(\exp(\frac{h\nu_{21}}{k_B T}) - 1\right)} \tag{12}$$

$$n_2 = \left(n_1 \frac{g_2}{g_1} - n_2\right) \frac{1}{\left(\exp(\frac{h\nu_{21}}{k_B T}) - 1\right)} \tag{13}$$

$$n_2 (1 + \frac{1}{\left(\exp(\frac{h\nu_{21}}{k_B T}) - 1\right)}) = n_1 \frac{g_2}{g_1} \frac{1}{\left(\exp(\frac{h\nu_{21}}{k_B T}) - 1\right)} \tag{14}$$

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left(-\frac{h\nu_{21}}{k_B T}\right), \tag{15}$$

where we obtain the typical "definition of the spin temperature":

$$\boxed{\frac{n_1}{n_0} = \frac{g_1}{g_0} \exp(-T_*/T_S) = 3 \exp(-0.068\mathrm{K}/T_S)} \tag{16}$$

where we define $T_* \equiv h\nu_{21}/k_B$. Note that the above relation also follows trivially from assuming that n_i is Boltzmann-distributed. We also did not apply the Rayleigh-Jeans approximation either.

Going back to the radiative transfer equation 1 but with our effective emissivities as introduced in equation 4, we introduce the optical depth $\tau_\nu = \int \mathrm{d}s \alpha_\nu^{\mathrm{eff}}$ with $\mathrm{d}\tau_\nu = \alpha_\nu^{\mathrm{eff}} \mathrm{d}s$:

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\mathrm{eff}} I_\nu + j_\nu^{\mathrm{eff}} \tag{17}$$

$$\frac{dI_\nu}{\alpha_\nu^{\mathrm{eff}} ds} \alpha_\nu^{\mathrm{eff}} = -\alpha_\nu^{\mathrm{eff}} I_\nu + j_\nu^{\mathrm{eff}} \tag{18}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{j_\nu^{\mathrm{eff}}}{\alpha_\nu^{\mathrm{eff}}} \tag{19}$$

$$\frac{dI_\nu}{d\tau_\nu} \equiv -I_\nu + S_\nu, \tag{20}$$

where in the last line we define the source function S_ν. We want to solve this equation. The first step is to perform a change of variables $I \equiv I_\nu \exp(\tau_\nu)$ and $S \equiv S_\nu \exp(\tau_\nu)$, such that:

$$\frac{d(I \exp(-\tau_\nu))}{d\tau_\nu} = -I \exp(-\tau_\nu) + S \exp(-\tau_\nu) \tag{21}$$

$$\frac{dI}{d\tau} \exp(-\tau_\nu) - I \exp(-\tau_\nu) = -I \exp(-\tau_\nu) + S \exp(-\tau_\nu) \tag{22}$$

$$\frac{dI}{d\tau_\nu} = S. \tag{23}$$

[2] Here we can either first get the Einstein relations by assuming the Boltzmann distribution for n_i in which case the spin temperature equation is trivial or assume the Einstein relations in order to "derive" the spin temperature equation without explicitly assuming the form of n_i. In other words, the spin temperature equation just follows from the Boltzmann distribution for n_i which is assumed in the Einstein relations.

Integrating and reversing the change of variables we obtain:

$$I(\tau_\nu) = I(0) + \int_0^{\tau_\nu} \mathrm{d}\tau' S(\tau') \tag{24}$$

$$I_\nu \exp(\tau_\nu) = I(0) + \int_0^{\tau_\nu} \mathrm{d}\tau' S_\nu \exp(\tau') \tag{25}$$

$$I_\nu(\tau_\nu) = \exp(-\tau) \left(I_\nu(0) + \int_0^{\tau_\nu} \mathrm{d}\tau' S_\nu \exp(\tau') \right) \tag{26}$$

and from this we obtain the familiar expression for radiative transfer over a uniform slab of IGM gas with constant α_ν and j_ν over the path s or τ_ν thus taking S_ν out of the integral:

$$\boxed{I_\nu(\tau_\nu) = I_\nu(0)\exp(-\tau_\nu) + S_\nu(1 - \exp(-\tau_\nu))} \tag{27}$$

Since the 21-cm line redshifts to low frequencies, the Rayleigh-Jeans limit is an excellent approximation to the Planck black-body curve so we take $I_\nu = \frac{2k_B T \nu^2}{c^2}$. Also assume IGM is in LTE and thus take that the source term $S_\nu = \frac{j_\nu}{\alpha_\nu} \propto T_S$. Plugging this into the above equation, $\frac{2k_B \nu^2}{c^2}$ cancel out from each side and we are left with:

$$\boxed{T_b = T_\gamma \exp(-\tau_\nu) + T_\mathrm{S}(1 - \exp(-\tau_\nu))} \tag{28}$$

2 21-cm Optical depth

Our next task is to solve for the optical depth $\tau_\nu = \int \mathrm{d}s\ \alpha_\nu^{\mathrm{eff}}$. From Equation 4, applying the Einstein relations and the definition of the spin temperature, we obtain:

$$\begin{aligned}
\alpha_\nu^{\mathrm{eff}} &= (n_1 B_{12} - n_2 B_{21}) h\nu_{21}\phi(\nu) && (29)\\
&= \left(n_1 \frac{g_2}{g_1} B_{21} - n_1 \frac{g_2}{g_1} \exp\left(-\frac{T_*}{T_S}\right) B_{21} \right) h\nu_{21}\phi(\nu) && (30)\\
&= n_1 \frac{g_2}{g_1} B_{21} h\nu_{21}\phi(\nu) \left(1 - \exp\left(-\frac{T_*}{T_S}\right)\right) && (31)\\
&= n_1 \frac{g_2}{g_1} \frac{c^2 A_{21}}{2\nu_{21}^2} \left(1 - \exp\left(-\frac{T_*}{T_S}\right)\right). && (32)
\end{aligned}$$

Assuming that the emissivity is isotropic, we multiply $\alpha_\nu^{\mathrm{eff}}$ by a factor of 4π and the absorption optical depth can be written as:

$$\tau_\nu = \int \mathrm{d}s\ \sigma_1(\nu) n_1 \phi(\nu)(1 - \exp(-T_*/T_S)), \tag{33}$$

where $\sigma_1(\nu) \equiv \frac{3c^2 A_{21}}{8\pi\nu_{21}^2}$, where $g_2/g_1 = 3$, is the absorption cross-section with the Einstein coefficient $A_{21} = 2.85 \times 10^{-15}\mathrm{s}^{-1}$ for the spontaneous decay rate of the spin-flip transition, $n_1 = \frac{n_{\mathrm{HI}}}{3\exp(-T_*/T_S)+1} \approx n_{\mathrm{HI}}/4$ in the limit of $T_* \ll T_S$.

$$\begin{aligned}
n_{\mathrm{HI}} &= n_{\mathrm{H}} x_{\mathrm{HI}}(1+\delta) && (34)\\
&= \frac{\Omega_{b,0}}{m_H} \rho_{c,0}(1+z)^3 x_{\mathrm{HI}}(1+\delta) && (35)\\
&= \frac{\Omega_{b,0}}{m_H} \frac{3H_0^2}{8\pi G}(1+z)^3 x_{\mathrm{HI}}(1+\delta) && (36)
\end{aligned}$$

Since $H(z) = H_0\sqrt{\Omega_{m,0}(1+z)^3}$ during matter domination:

$$n_{\mathrm{HI}} = x_{\mathrm{HI}}(1+\delta) \frac{3\Omega_{b,0} H_0}{8\pi G m_H} \frac{H(z)}{\sqrt{\Omega_{m,0}}} (1+z)^{3/2} \tag{37}$$

$\phi(\nu)$ is the normalised 21-cm line profile such that $\int \phi(\nu)\mathrm{d}\nu = 1$. The line profile may include effects such as natural broadening due to the finite lifetime of excited states, pressure broadening due to collisions between the atoms, and bulk motion due to large-scale peculiar velocities. The line profile for the 21-cm line is narrow and can be well-approximated by a Dirac delta function at the rest-frame frequency $\phi(\nu) \approx \delta(\nu - \nu_{21})$. Evaluating the integral, we obtain the expression for the 21-cm optical depth at observed frequency $\nu_{\rm obs}$ (so not rest-frame) with $T_* \ll T_S$ so that $1 - \exp(-T_*/T_S) \approx T_*/T_S = 2\pi\hbar\nu_{21}/(k_B T_S)$ and $\mathrm{d}s = -\mathrm{d}\nu \frac{c}{H(z)\nu_{21}}\left(1 + \frac{\mathrm{d}v_r/\mathrm{d}r}{H(1-v_r/c)}\right)^{-1}$ converts the integral from distance s to observed frequency accounting for Doppler shift:

$$\tau_\nu \approx \int \mathrm{d}s\, \frac{3c^2 A_{21}}{32\pi\nu_{21}^2} x_{\rm HI}(1+\delta)\frac{3\Omega_{b,0}H_0}{8\pi G m_H}\frac{H(z)}{\sqrt{\Omega_{m,0}}}(1+z)^{3/2}\delta(\nu-\nu_{21})(1-\exp(-T_*/T_S)) \tag{38}$$

$$\tau_{\nu_{\rm obs}} \approx \frac{9\hbar c^3}{32\pi\nu_{21}^2}\frac{A_{21}}{4k_B}\frac{\Omega_{b,0}h}{m_H G\sqrt{\Omega_{m,0}}} x_{\rm HI}(1+\delta)\left(\frac{1+z}{10}\right)^{1/2}\left(\frac{10}{T_S}\right)\left(\frac{H}{H+\mathrm{d}v_r/\mathrm{d}r}\right) \tag{39}$$

$$\approx 0.003\, x_{\rm HI}(1+\delta)\left(\frac{1+z}{10}\right)^{1/2}\left(\frac{10\ \mathrm{K}}{T_S}\right)\left(\frac{H}{H+\mathrm{d}v_r/\mathrm{d}r}\right) \tag{40}$$

3 21-cm Brightness temperature

Under the assumption of a small optical depth, we obtain the 21-cm brightness temperature from equations 28 and 38:

$$\begin{aligned}
\delta T_{\rm b} &= T_b - T_\gamma && (41)\\
&= T_{\rm S}(1-\exp(-\tau_{\nu_{\rm obs}})) + T_\gamma \exp(-\tau_{\nu_{\rm obs}}) - T_\gamma && (42)\\
&= (T_{\rm S} - T_\gamma)\left(1 - e^{-\tau_{\nu_{\rm obs}}}\right) && (43)\\
&\approx (T_{\rm S} - T_\gamma)\,\tau_{\nu_{\rm obs}} && (44)\\
&\approx 35\, x_{\rm HI}(1+\delta)\left(\frac{\Omega_b h^2}{0.023}\right)\left(\frac{0.15}{\Omega_m h^2}\frac{1+z}{10}\right)^{1/2}\\
&\times\left(1 - \frac{T_\gamma}{T_{\rm S}}\right)\left(\frac{H}{H+\mathrm{d}v_r/\mathrm{d}r}\right)\ \mathrm{mK},
\end{aligned}$$

Code to compute the constants:

```
import numpy as np
from astropy.constants import G, c, k_B, h
import astropy.units as un

# define some extra constants:
# hydrogen mass
m_H = 1.6735e-24*un.g

# Einstein coefficient for spontaneous emission
A21 = 2.85e-15/un.s

# 21-cm rest-frame frequency
nu_21 = 1.420e9/un.s

# h little
h_little = 0.67

# Hubble constant
H0 = 100.*h_little*un.km/un.s/un.Mpc
h_little_w_units = H0/100.
Ob0 = 0.05
Om0 = 0.3

cst = 9*h*c**3*A21*10**1.5*Ob0*h_little_w_units/(32*np.pi**2*8*k_B*nu_21**2*m_H*G*np.sqrt(Om0))
```

```
print(cst.to("K")) # = 0.00365349 K

# This includes 10**1.5 for the 10 in redshift and T_S terms.
# To get the same as Furlanetto, Oh, Briggs review:
print(cst.to("K")*10**0.5) # 0.01155 ~ 0.0092

# To get the constant in front of \delta T_b, we need to include the factors from the cosmology t
print(cst.to("mK")*0.023/0.15**0.5/Ob0*np.sqrt(Om0)/h_little*10) # 35.47 mK
```